12.2 Vectors Intro

Goal: Introduce vector basics.

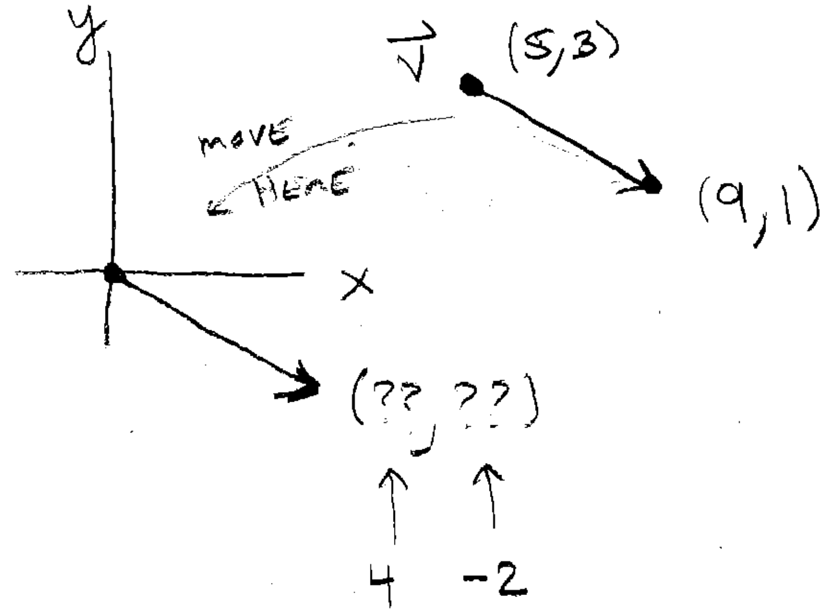
Def'n: A **vector** is a quantity with magnitude and direction.

We depict a vector with an arrow:

a. The length is the *magnitude*.
b. The 'tail' of the arrow is called the *initial point* and the 'head' is called the *terminal point.*

If the vector is drawn with the tail at the origin and that results in the head being at the point (v_1, v_2, v_3), then we denote the vector by

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle$$



$\vec{v} = \langle 4, -2 \rangle$

Basic fact list:

- Two vectors are equal if all components are equal.

$\langle 3, a\rangle = \langle b, 4\rangle \Rightarrow a = 4$ and $b = 3$

- We denote **magnitude** by

$$|\boldsymbol{v}| = \sqrt{{v_1}^2 + {v_2}^2 + {v_3}^2}$$

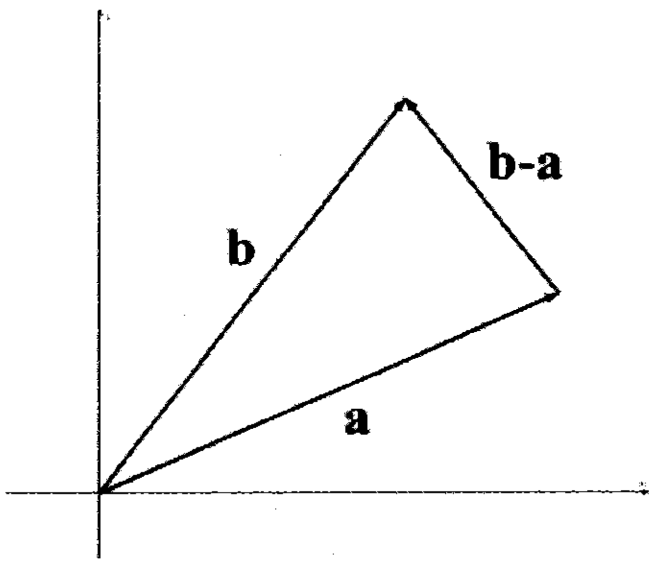
$$|\langle 3,-1,2\rangle| = \sqrt{(3)^2 + (-1)^2 + (2)^2} = \sqrt{9+1+4} = \sqrt{14}$$

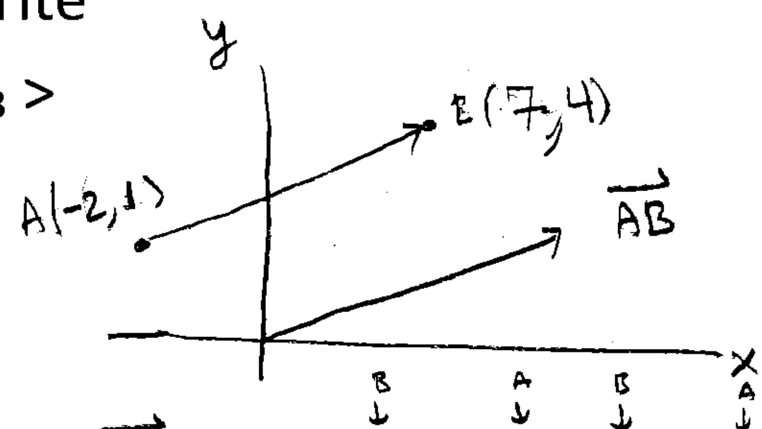
- To denote the **vector from $\mathbf{A(a_1,a_2,a_3)}$ to $\mathbf{B(b_1,b_2,b_3)}$,** we write

$$\overrightarrow{AB} = < b_1 - a_1,\ b_2 - a_2,\ b_3 - a_3 >$$

"DIRECTED LINE SEGMENT"

HW 2 & 3 AND MANY MANY OTHER PLACES





$$\overrightarrow{AB} = \langle \overset{B}{7} - \overset{A}{-2},\ \overset{B}{4} - \overset{A}{1}\rangle = \langle 9, 3\rangle$$

$$\overrightarrow{BA} = \langle -9, -3\rangle$$

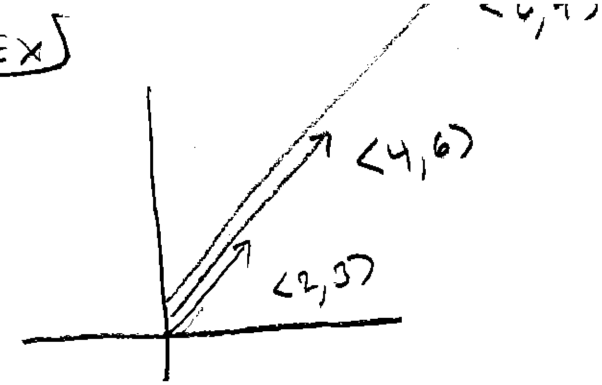
- **Scalar Multiplication**

If c is a constant, then we define

$$c\mathbf{v} = \langle cv_1, cv_2, cv_3 \rangle,$$

which scales the magnitude by a factor of c.

Ex)



$\vec{v} = \langle 2, 3 \rangle$

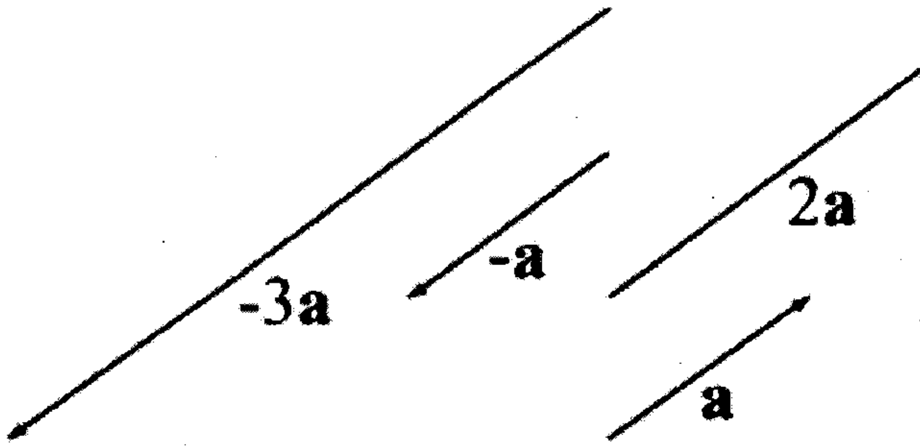
$2\vec{v} = \langle 4, 6 \rangle$

$3\vec{v} = \langle 6, 9 \rangle$

$-2\vec{v} = \langle -4, -6 \rangle$

$-\vec{v} = \langle -2, -3 \rangle$

DO NOT PUT ANY MULTIPLICATION SYMBOL HERE. (THAT WILL POSSIBLY MEAN SOMETHING ELSE LATER)



- A **unit vector** has length one.

Note:

$\frac{1}{|\boldsymbol{v}|}\boldsymbol{v}$ = "unit vector in the same direction as **v**".

IMPORTANT!

$|\vec{v}| = \sqrt{2^2 + 3^2} = \sqrt{13}$ ← NOT A UNIT VECTOR

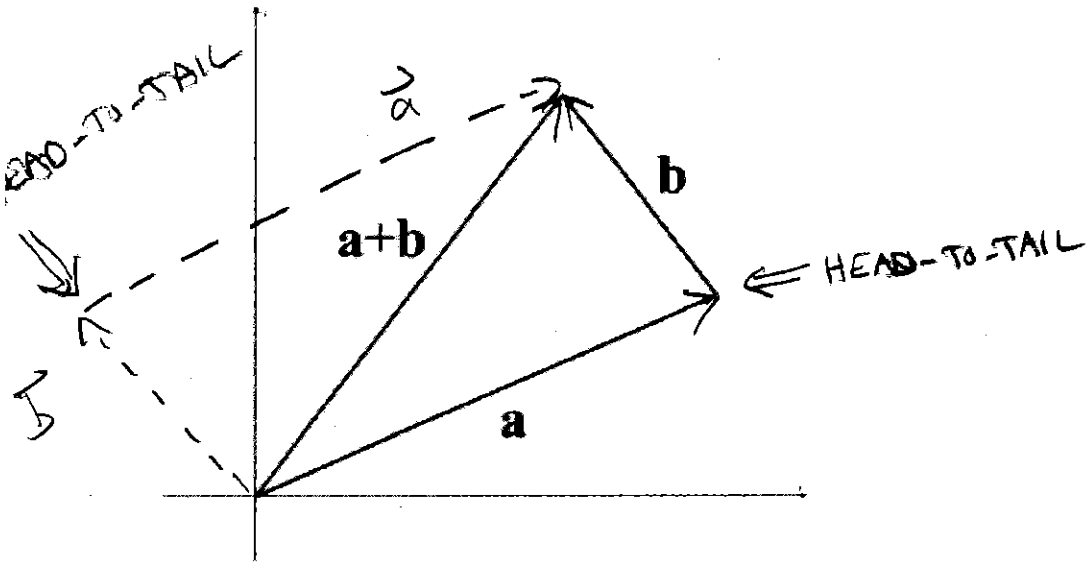
$$\frac{1}{|\vec{v}|}\vec{v} = \frac{1}{\sqrt{13}}\langle 2, 3 \rangle = \left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle$$

← UNIT VECTOR IN SAME DIRECTION AS $\vec{v}$!

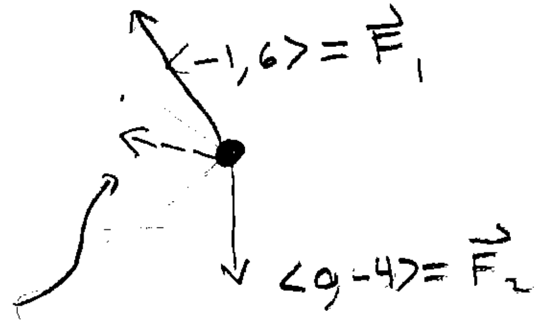
- We define the **vector sum** by

$$\mathbf{v} + \mathbf{w} = \langle v_1, v_2, v_3 \rangle + \langle w_1, w_2, w_3 \rangle$$
$$= \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle$$

$\vec{a} + \vec{b}$ = diagonal of the parallelogram formed by drawing $\vec{a}$ AND $\vec{b}$ HEAD-TO-TAIL.



Ex)



$$\vec{F}_1 + \vec{F}_2 = \langle -1, 6 \rangle + \langle 0, -4 \rangle$$
$$= \langle -1, 2 \rangle = \text{"RESULTANT FORCE"}$$

(OBJECT WILL ACCELERATE IN THIS DIRECTION)

- Standard unit basis vectors:

$\mathbf{i} = \langle 1, 0, 0 \rangle$

$\mathbf{j} = \langle 0, 1, 0 \rangle$

$\mathbf{k} = \langle 0, 0, 1 \rangle$

SAME!

$$3\vec{i} + 2\vec{j} - \vec{k}$$
$$= 3\langle 1,0,0 \rangle + 2\langle 0,1,0 \rangle - \langle 0,0,1 \rangle$$
$$= \langle 3, 2, -1 \rangle$$

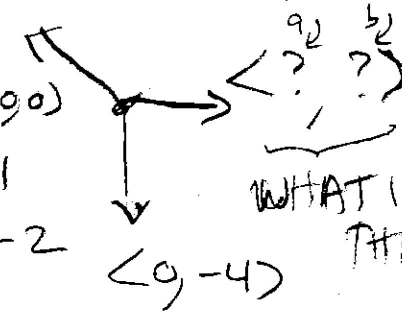
NOW ASSUME OBJECT IS NOT MOVING

$\langle -1, 6 \rangle$

$\langle 0,-4 \rangle + \langle -1, 6 \rangle + \langle a, b \rangle = \langle 0, 0 \rangle$

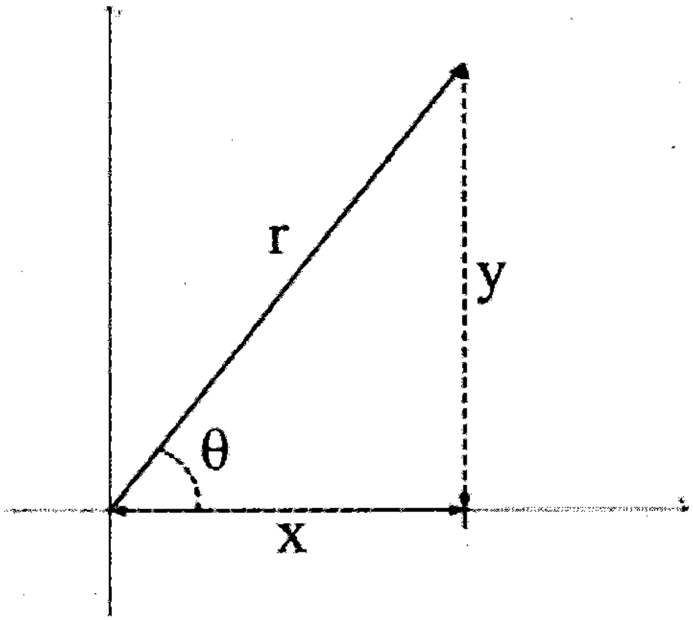
$$\Rightarrow \begin{cases} 0 - 1 + a = 0 \Rightarrow a = 1 \\ -4 + 6 + b = 0 \Rightarrow b = -2 \end{cases}$$

$\langle 1, -2 \rangle$



$\underbrace{\langle \overset{a}{?}, \overset{b}{?} \rangle}_{\text{WHAT IS THIS !??}}$

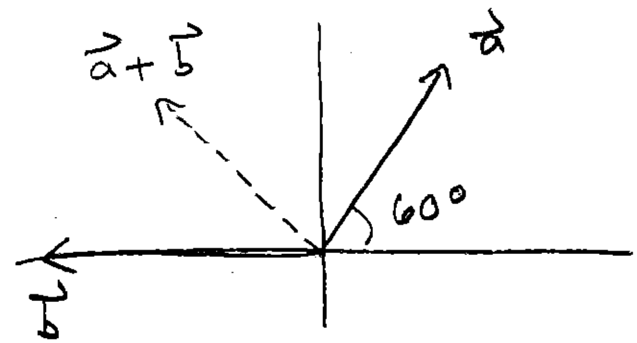
- In 2D, you may be given the angle, θ, and length, *r*, as shown



Remember,

$x = r\cos(\theta)$, $y = r\sin(\theta)$, $x^2 + y^2 = r^2$.

FROM HOMEWORK



GIVEN $|\vec{a}| = 200$ N

$|\vec{b}| = 300$ N

FIND $\vec{a}+\vec{b}$

$\boxed{1}$ $\vec{b} = \langle -300, 0 \rangle$

$\boxed{2}$ $\vec{a} = \langle 200\cos(60°), 200\sin(60°) \rangle$

$= \langle 200 \cdot \frac{1}{2}, 200 \cdot \frac{\sqrt{3}}{2} \rangle$

$= \langle 100, 100\sqrt{3} \rangle$

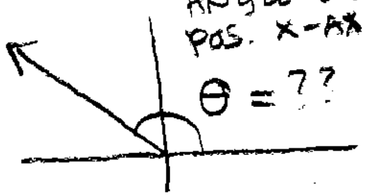
$\boxed{3}$ $\vec{a}+\vec{b} = \langle 100, 100\sqrt{3} \rangle + \langle -300, 0 \rangle$

$= \langle -200, 100\sqrt{3} \rangle =$ Resultant Force

$\boxed{4}$

ANGLE WITH POS. X-AXIS ??

$\theta = ??$



$|\vec{a}+\vec{b}| = \sqrt{(-200)^2 + (100\sqrt{3})^2} = \sqrt{40000 + 10000 \cdot 3}$

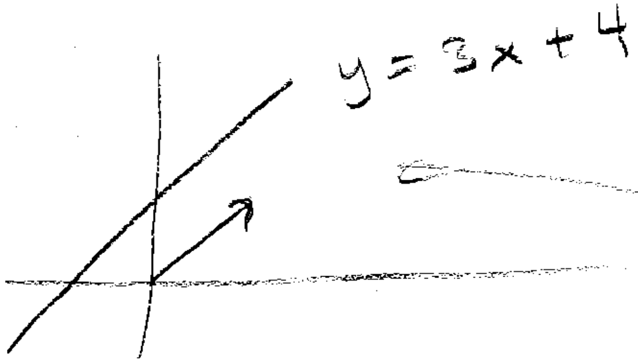
$= \sqrt{70000}$

$= 100\sqrt{7} \approx 264.575$ N

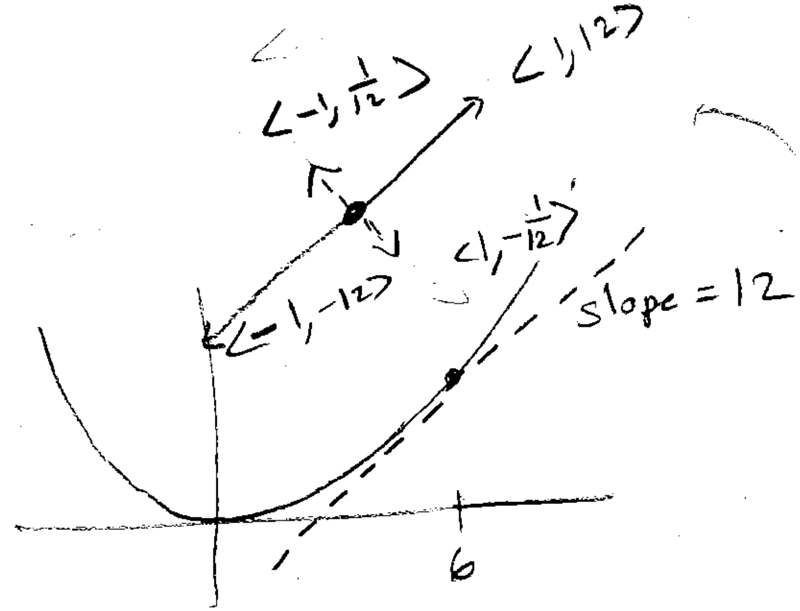
$-200 = r\cos\theta$

$\Rightarrow \cos\theta = \frac{-200}{264.575} \Rightarrow \theta = \cos^{-1}\left(\frac{-200}{264.575}\right) \approx 139.1°$

- In 2D, if you want a vector that is **parallel to a line with slope *m*,** then the vector < 1 , m > works.



$\vec{v} = \langle 1, 3 \rangle$ is parallel to this line



Ex) $y = x^2$ FIND A VECTOR PARALLEL TO THE TANGENT LINE AT THE POINT (6, 36)

$y' = 2x \Rightarrow y'(6) = 12 =$ "slope of tangent line"

$\vec{v} = \langle 1, 12 \rangle$ is parallel to the tangent line

ALSO $-\vec{v} = \langle -1, -12 \rangle$ is parallel

NOTE: SLOPE OF PERPENDICULAR IS $-\frac{1}{12}$. SO $\langle 1, -\frac{1}{12} \rangle$, $\langle -1, \frac{1}{12} \rangle$ ARE PERPENDICULAR TO THE ...

12.3 Dot Products

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and

$\mathbf{b} = \langle b_1, b_2, b_3 \rangle$

Then we define the dot product by:

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Ex) $\vec{a} = \langle 3,1,2 \rangle$

$\vec{b} = \langle -1,6,5 \rangle$

$\vec{a} \cdot \vec{b} = (3)(-1) + (1)(6) + (2)(5)$

$= -3 + 6 + 10$

$= 13$ ← A NUMBER ??

$\boxed{\text{NOTE}}$

$$\underset{\text{VECTOR}}{\vec{a}} \bullet \underset{\text{VECTOR}}{\vec{b}} = \underset{\text{NUMBER}}{c}$$

Basic fact list:

- Manipulation facts (like regular multiplication):

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

SEE ABOVE $\vec{b} \cdot \vec{a} = 13$

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

SAME

$\langle 2,3 \rangle \cdot (\langle 1,1 \rangle + \langle 3,4 \rangle)$

$= \langle 2,3 \rangle \cdot \langle 1,1 \rangle + \langle 2,3 \rangle \cdot \langle 3,4 \rangle$

$$c(\mathbf{a} \cdot \mathbf{b}) = (c\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (c\mathbf{b})$$

$2 \; \langle 4,3 \rangle \cdot \langle -1,3 \rangle$

$= 2 \cdot (-4 + 9) = 2 \cdot 5 = 10$

$\langle 8,6 \rangle \cdot \langle -1,3 \rangle = -8 + 18$

$(2\vec{a}) \cdot \vec{b} = 10$

$$(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = ???$$

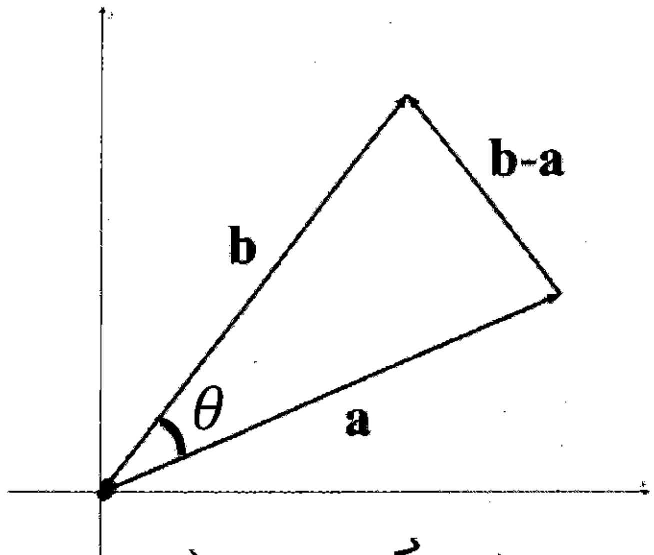
$\vec{a} \cdot \vec{a} + 2\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b}$

- Helpful fact:

$$\mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + a_3^2 = |\boldsymbol{a}|^2$$

Most *important* dot product fact:

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\theta)$$



NOTE: $\vec{a}$ AND $\vec{b}$ ARE TAIL-TO-TAIL!

θ = "ANGLE BETWEEN WHEN DRAWN TAIL-TO-TAIL"

Proof (not required):

(1) By the Law of Cosines:

$$|\mathbf{b}-\mathbf{a}|^2 = |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos(\theta)$$

(2) The left-hand side expands to

$$\begin{aligned}|\mathbf{b}-\mathbf{a}|^2 &= (\mathbf{b}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{a})\\ &= \mathbf{b}\cdot\mathbf{b} - 2\mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{a}\\ &= |\mathbf{b}|^2 - 2\mathbf{a}\cdot\mathbf{b} + |\mathbf{a}|^2\end{aligned}$$

Subtracting $|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2$ from both sides of (1) yields:

$$-2\mathbf{a}\cdot\mathbf{b} = -2|\mathbf{a}||\mathbf{b}|\cos(\theta).$$

Divide by -2 to get the result. (QED)

Ex) $\vec{a} = \langle 1,1\rangle \quad \vec{b} = \langle 0,4\rangle$

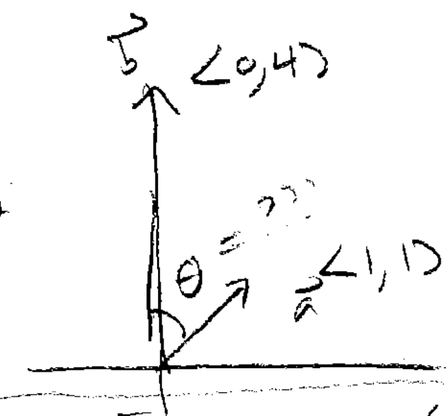
$|\vec{a}| = \sqrt{2} \qquad |\vec{b}| = 4$

$\vec{a}\cdot\vec{b} = 0 + 4 = 4$

$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$\underbrace{\quad}_{4} = \sqrt{2}\ 4\cos\theta$

$\cos\theta = \frac{1}{\sqrt{2}} \leftarrow \frac{\sqrt{2}}{2}$



$\boxed{\theta = \pi/4 \text{ rad } (45^\circ)}$

Most important consequence:

If **a** and **b** are orthogonal, then

$$\mathbf{a}\cdot\mathbf{b} = 0$$

Ex)

$\vec{a} = \langle 1, 1, 2\rangle$

$\vec{b} = \langle 3, 2, -5\rangle$

$\vec{c} = \langle -6, 4, 1\rangle$

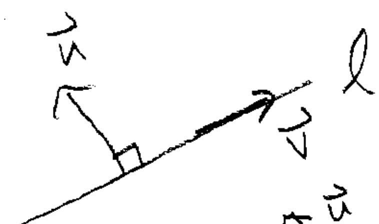
ARE ANY OF THESE ORTHOGONAL?

$\vec{a}\cdot\vec{b} = 3 + 2 - 10 = -5$ NO

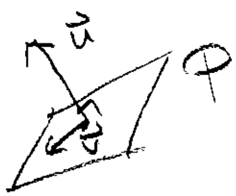
$\vec{a}\cdot\vec{c} = -6 + 4 + 2 = 0$ YES

$\vec{b}\cdot\vec{c} = -18 + 8 - 5 = -15$ NO

ASIDE



$\vec{u}\cdot\vec{v} = 0.$



$\vec{u}\cdot\vec{v} = 0$

Also:

If **a** and **b** are parallel, then

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|$$

or

$$\mathbf{a}\cdot\mathbf{b} = -|\mathbf{a}||\mathbf{b}|$$

EASIER TO CHECK IF THEY ARE CONSTANT MULTIPLES. $\cdot 4$ for ALL components

$\langle 1, 3, -5\rangle$ AND $\langle 4, 12, -20\rangle$

ARE PARALLEL!

$\langle 1, 3, -5\rangle$ AND $\langle 4, 12, -17\rangle$

ARE NOT PARALLEL! NOT TIMES 4.

Projections:

NEXT TIME!